1. Let $G$ be a group in which $(ab)^m = a^m b^m$ for three consecutive integers and for all $a, b \in G$. Prove that $G$ is abelian.

2. Prove that the subgroup $N$ of $G$ is a normal subgroup of $G$ if and only if every left coset of $N$ in $G$ is a right coset of $N$ in $G$.

3. Prove that a finite integral domain is a field.

4. If $U, V$ are ideals of $R$, let $U + V = \{u + v : u \in U, v \in V\}$. Prove that $U + V$ is also an ideal of $R$. 
5. If $V$ is a finite-dimensional space over $F$, prove that any two bases of $V$, have the same number of elements.

6. If $V$ is a vector space and $u, v \in V$, then prove that $|\langle u, v \rangle| \leq \|u\|\|v\|$.

7. If $L$ is an algebraic extension of $K$ and if $K$ is an algebraic extension of $F$, then prove that $L$ is an algebraic extension of $F$.

8. If $V$ is finite-dimensional over $F$, prove that $T \in A(V)$ is regular if and only if $T$ maps $V$ onto $V$.

SECTION B — (5 × 10 = 50 marks)

Answer any FIVE questions.

9. State and prove first part of Sylow’s theorem.

10. State and prove Cayley’s theorem.

11. If $R$ is a ring with unit element, then for all $a, b \in R$ prove that

(a) $a.0 = 0.a = 0$

(b) $a(-b) = (-a)b = -(ab)$

(c) $(-a)(-b) = ab$

(d) $(-1)a = -a$

(e) $(-1)(-1) = 1$. 

2 PG–387
12. Prove that every integral domain can be imbedded is a field.

13. If \( v_1, v_2, \ldots, v_n \) is a basis of \( V \) over \( F \) and if \( w_1, w_2, \ldots, w_m \) in \( V \) are linearly independent over \( F \), prove that \( m \leq n \).

14. If \( V \) and \( W \) are of dimensions \( m \) and \( n \) respectively over \( F \), then prove that \( \text{Hom}(V,W) \) is of dimensions \( mn \) over \( F \).

15. If \( F \) is of characteristic 0 and if \( a, b \) are algebraic over \( F \), then prove that there exist an element \( c \in F(a,b) \) such that \( F(a,b) = F(c) \).

16. If \( T \in A(V) \) has all its characteristic roots is \( F \), then prove that there is a basis of \( V \) is which the matrix of \( T \) is triangular.
M.Sc. DEGREE/P.G. DIPLOMA

First Year
Mathematics

REAL ANALYSIS

Time : 3 hours Maximum marks : 75

SECTION A — (5 \times 5 = 25 marks)

Answer any FIVE questions.

1. Prove that the ordered set \( R \) has the least upper bound property.

2. Prove that compact subsets of metric spaces are closed.

3. Prove that if \( f \) is continuous at a point \( p \in E \) and if \( g \) is continuous at \( f(p) \) then prove that \( h = g \circ f \) is continuous at \( p \).
4. Let \( f \) be monotonic on \((a, b)\), then prove that the set of points of \((a, b)\) at which \( f \) is discontinuous is almost countable.

5. State and prove mean value theorem.

6. If \( p^* \) is a refinement of \( p \) then prove that
\[ U(p^*, f, \alpha) \leq U(p, f, \alpha). \]

7. State and prove Weierstrass theorem.

8. Prove that a linear operator \( A \) on \( R^n \) is invertible if and only if \( \det[A] = 0 \).

SECTION B — (5 \times 10 = 50 marks)

Answer any FIVE questions.

9. Prove that for every real \( x > 0 \) and every integer \( n > 0 \) there is one and only one real \( y \) such that \( y^n = x \).

10. Prove that
\[ \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e. \]

11. Let \( f \) be a continuous mapping of a compact metric space \( X \) into a metric space \( Y \), then prove that \( f \) is uniformly continuous on \( X \).

12. State and prove L’Hospital rule.
13. If $\gamma'$ is continuous on $[a, b]$, then prove that $\gamma$ is rectifiable and $\Lambda(r) = \int_{r}^{b} |\gamma'(t)| \, dt$.


15. State and prove Parseval’s theorem.

16. State and prove the contraction principle.
P.G. DIPLOMA IN MATHEMATICS
EXAMINATION — JUNE 2018.

TOPOLOGY AND FUNCTIONAL ANALYSIS

Time : 3 hours Maximum marks : 75

PART A — (5 × 5 = 25 marks)

Answer any FIVE questions.

1. If \( Y \) is a subspace of \( X \), then prove that set \( A \) is closed in \( Y \) if and only if it equals the intersection of a closed set of \( X \) with \( Y \).

2. Prove that the image of a connected space under a continuous map is connected.

3. Prove that compactness implies the limit point compactness but not conversely.

4. Define the following:
   (a) First countable space
   (b) Second countable space
   (c) Dense subset.
5. Prove that the space $l^n_p$ of all $n$-tuples $x = (x_1, x_2, \ldots, x_n)$ of scalars with the norm defined by $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$ is a Banach space.

6. State and prove the uniform Boundedness theorem.

7. State and prove the Schwarz inequality in a Hilbert space $H$.

8. If $M$ is a closed linear subspace of a Hilbert space $H$, prove that $H = M \oplus M^\perp$.

PART B — (5 × 10 = 50 marks)

Answer any FIVE questions.

9. If $f$ is a continuous function from $X \rightarrow Y$, then prove that for every convergent sequence $\{x_n\} \rightarrow x$ in $X$, the sequence $\{f(x_n)\}$ converges to $f(x)$ in $Y$. Show also that the converse holds if $X$ is metrizable.

10. Show that finite Cartesian product of connected spaces is connected.

2

PG–455
11. Let $X$ be a non-empty Hausdorff space. If $X$ has no insolated points, then prove that $X$ is uncountable.

12. Suppose that $X$ has a countable basis. Then prove the following:

   (a) Every open covering of $X$ contains a countable subcollection covering $X$.

   (b) There exists a countable subset of $X$ that is dense in $X$.

13. Let $M$ be a closed linear subspace of a normed linear space $N$. If the norm of a coset $x + M$ in the quotient space $N/M$ is defined by $\|x + M\| = \inf \{\|x + m\| / m \in M\}$, then prove that $N/M$ is a normed linear space. If $N$ is a Banach space, then prove that $N/M$ a Banach space.

14. State and prove the closed graph theorem.

15. Let $H$ be a Hilbert space and let $f$ be an arbitrary functional in $H^*$. Then prove that there exists a unique vector $y$ in $H$ such that $f(x) = (x, y)$ for every $x$ in $H$. 

3
16. (a) If $T$ is an operator on $H$ then prove that $T$ is normal $\iff$ its real and imaginary parts commute.

(b) Prove that an operator $T$ on $H$ is unitary $\iff$ it is an isometric isomorphism of $H$ onto itself.